

# Exactly Solvable Potentials by $\mathcal{SO}(2,2)$ Dynamical Algebra

S.-A. Yahiaoui, M. Bentaiba<sup>1</sup>

LPTHIRM, Département de Physique, Faculté des Sciences,  
Université Saad DAHLAB de Blida, Algeria.

## Abstract

The differential realization of the potential group  $\mathcal{SO}(2,2)$  is used. The spectrum-generating algebra for a kind of exactly solvable potentials endowed with position-dependent mass is constructed.

Keywords : Lie algebras, Casimir invariant operators, Supersymmetry, Spectrum-generating algebras.

PACS : 02.20.-a; 02.20.Qs; 02.20.Sv; 03.65.Fd; 03.65.Ca

## 1 Introduction

Exact solutions for some quantum mechanical systems endowed with position-dependent effective mass have attracted, in recent years, much attention on behalf of physicists [1-10]. Effective mass Schrödinger equation was introduced by BenDaniel and Duke [1] in order to explain the behavior of electrons in semiconductors. It have also many applications in the fields of material sciences and condensed matter physics such as quantum well and quantum dots [11],  $^3H$ -clusters [12], quantum liquids [13], graded alloys [14], and heterostructures [15], etc.

In theoretical physics, various variety of methods and approaches are employed for generating exactly solvable one-dimensional potentials such as the supersymmetric quantum mechanical method [16] and group theory

---

<sup>1</sup>Corresponding author:

E-mail address : bentaiba@hotmail.com  
bentaiba1@caramail.com  
bentaiba@mail.univ-blida.dz

through the Lie algebra approach [17-20]. Both of these two approaches give (almost) identical bound-state energy eigenvalues. Supersymmetric quantum mechanics have been recognized as the reformulation of the factorization method; i.e. the generalization of the creation and annihilation operators given two isospectral Hamiltonians, while Lie algebras provides us with an important aspect from algebraic techniques which are used to construct the Hamiltonian(s) from the Casimir operator(s) related to the group-algebraic structures.

A different notion about spectrum-generating algebra (SGA) techniques, not totally independent of previous concepts [22], was introduced by Cordero et al.[23-26], and are based on the use of a realization of the infinitesimal generators  $\mathcal{J}_i$  ( $i = 0, +, -$ ) in the non-compact group  $\mathcal{SO}(2, 1)$ . Following their definition, the generators of SGA can be used to replace the canonical variables in the Schrödinger equation in such a way that the bound-state energy spectrum is connected to the spectrum of a compact generators of the group [26]. A convenient way to construct a SGA for physical systems is by introducing a set of boson creation and annihilation operators [17], these later can be recast into a set of generators of the underlying group. In terms of these settings, the non-compact algebra appears either as a spectrum-generating algebra or as a construction identified as potential algebra.

In the present article, we provide an extension of the algebraic treatments within the context of the non-compact group  $\mathcal{SO}(2, 2)$  using SGA to the description of the bound-state problems by connecting the Hamiltonian of the supersymmetric quantum mechanics to that of the powerful machinery of the group theory. It seems that considering the larger  $\mathcal{SO}(2, 2)$  potential algebra and applying the SGA scheme to them could be instructive to recover a wide kind of exactly solvable (confluent) Natanzon potentials defined by Natanzon [27]. With this, we generate and "rediscover" all potentials generated with a similar differential realization of the  $\mathfrak{su}(1, 1) \simeq \mathfrak{so}(2, 1)$  algebra [7,21,23,24]. It is well-known that the general families of the exactly solvable potentials can be categorized in two different families : the Natanzon and Natanzon confluent potentials. Each potentials in the same family can be mapped into each others by applying point canonical transformation (PCT) [9,10]. However, the link between

potentials belonging to the two different families is concerned with transformations larger than PCT. The required transformations are identified as being integral transformations [17,28].

The organization of the present article is as follows. In section 2 we introduce a specific differential realization of the  $\mathfrak{so}(2, 2)$  algebra inspired by the potential-group method and derived from the Hamiltonians which are expressed in terms of the Casimir operators of the same algebra. We present and define also the general formulation of the Hamiltonians endowed with position-dependent effective mass in the framework of supersymmetric quantum mechanics. The main results are contained in section 3, where the general expression of the effective potential is deduced once the Hamiltonians of the previous sections are combined. In section 4, we summarize our results by choosing the appropriate values for the parameters  $b$  and  $q$  thus leading to generate quantum mechanical effective potentials. Finally, the last section is devoted to some remarks and a conclusion.

## 2 Differential realization of the $\mathfrak{so}(2, 2)$

According to Lévai et al.[20], we start by giving a differential realization of the  $\mathfrak{so}(n, m)$  Lie algebras with six-generators containing altogether ten functions. To be more precise, these differential realizations are related to the algebras of  $\mathfrak{so}(2, 2)$ ,  $\mathfrak{so}(4)$  and  $\mathfrak{so}(3) \oplus \mathfrak{so}(2, 1)$ , allowing a more general form of the generators divided into two sets

$$\mathcal{J}_{\pm} = e^{\pm i\phi} [\pm h_1(x) \partial_x \pm g_1(x) + f_1(x) \mathcal{J}_0 + c_1(x) + k_1(x) \mathcal{L}_0], \quad (1.a)$$

$$\mathcal{J}_0 = -i\partial_{\phi}, \quad (1.b)$$

$$\mathcal{L}_{\pm} = e^{\pm i\chi} [\pm h_2(x) \partial_x \pm g_2(x) + f_2(x) \mathcal{L}_0 + c_2(x) + k_2(x) \mathcal{J}_0], \quad (1.c)$$

$$\mathcal{L}_0 = -i\partial_{\chi}, \quad (1.d)$$

where we have used the abbreviation  $\partial_{\Sigma} = \frac{d}{d\Sigma}$ , with  $\Sigma = x, \phi, \chi$ . Then the commutation relations are given by

$$[\mathcal{J}_0, \mathcal{J}_{\pm}] = \pm \mathcal{J}_{\pm} \quad ; \quad [\mathcal{J}_+, \mathcal{J}_-] = -2a\mathcal{J}_0, \quad (2.a)$$

$$[\mathcal{L}_0, \mathcal{L}_{\pm}] = \pm \mathcal{L}_{\pm} \quad ; \quad [\mathcal{L}_+, \mathcal{L}_-] = -2b\mathcal{L}_0, \quad (2.b)$$

$$[\mathcal{J}_i, \mathcal{L}_j] = 0, \quad (i, j = 0, +, -). \quad (2.c)$$

where  $a, b = \pm 1$ . For  $a = b = 1$ , we get  $\mathfrak{so}(2, 2) \simeq \mathfrak{so}(2, 1) \oplus \mathfrak{so}(2, 1)$  algebra, while for  $a = -b = 1$  and  $a = b = -1$  we obtain, respectively, the  $\mathfrak{so}(3) \oplus \mathfrak{so}(2, 1)$  and  $\mathfrak{so}(4) \simeq \mathfrak{so}(3) \oplus \mathfrak{so}(3)$  algebras. The above equations lead to a system of first-order equations for the functions appearing in (1.a) and (1.c) [20]

$$k_2^2 - h_2 k_2' = 0; \quad h_2 f_2' - f_2 k_2 = 0; \quad k_2^2 - f_2^2 = b, \quad (3.a)$$

$$h_1 = A h_2; \quad f_1 = A k_2; \quad k_1 = A f_2; \quad g_1 = A g_2, \quad (3.b)$$

$$c_1 = c_2 = 0, \quad (3.c)$$

with  $A^2 ab = 1$  and  $b^2 = 1$ . Here, we assume that  $h_i(x) \neq 0$ ,  $k_i(x) \neq 0$  and  $f_i(x) \neq 0$ , with  $i = 1, 2$ . It is obvious, from (3.a), that the choice of the function  $h_2(x)$  determines completely the shape of the functions  $f_2(x)$  and  $k_2(x)$ .

An interesting way to solve the differential equations (3.a) is by applying a variable transformation  $x \rightarrow y \equiv y(x)$  which changes the functions  $h_2(x)$ ,  $f_2(x)$  and  $k_2(x)$  into the following forms [21]

$$h_2(x) \rightarrow \tilde{h}_2(x) = h_2(x) \frac{dy(x)}{dx}, \quad (4.a)$$

$$k_2(x) \rightarrow \tilde{k}_2(x) = k_2(x), \quad (4.b)$$

$$f_2(x) \rightarrow \tilde{f}_2(x) = f_2(x), \quad (4.c)$$

$$g_2(x) \rightarrow \tilde{g}_2(x) = g_2(x), \quad (4.d)$$

$$c_2(x) \rightarrow \tilde{c}_2(x) = c_2(x). \quad (4.e)$$

It is clear that the operators in (1.a) and (1.c) are maintained invariants once the transformations (4) are applied. Now assuming that  $h_2(x) \frac{dy(x)}{dx} = \sqrt{b}y(x)$ , the first two differential equations in (3.a) become

$$\sqrt{b}y(x) f_2'(x) - k_2(x) f_2(x) = 0, \quad (5.a)$$

$$k_2^2(x) - \sqrt{b}y(x) k_2'(x) = 0, \quad (5.b)$$

where their solutions are given, respectively, by

$$k_2(x) = \sqrt{b} \frac{1 + qy^2(x)/\sqrt{b}}{1 - qy^2(x)/\sqrt{b}}, \quad f_2(x) = \frac{\delta y(x)}{1 - qy^2(x)/\sqrt{b}}, \quad (6)$$

where  $q$  and  $\delta$  are constants of integration.

Since the  $\mathfrak{so}(2, 2)$  algebra is of rank 2, then it admits two Casimir operators given by

$$\mathcal{C}_{\mathfrak{so}(2,2)}^{(\mathcal{J}, \mathcal{L})} = 2\mathcal{C}_{\mathfrak{so}(2,1)}^{(\mathcal{J})} + 2\mathcal{C}_{\mathfrak{so}(2,1)}^{(\mathcal{L})}, \quad (7.a)$$

$$\tilde{\mathcal{C}}_{\mathfrak{so}(2,2)}^{(\mathcal{J}, \mathcal{L})} = 2\mathcal{C}_{\mathfrak{so}(2,1)}^{(\mathcal{J})} - 2\mathcal{C}_{\mathfrak{so}(2,1)}^{(\mathcal{L})}, \quad (7.b)$$

A basis  $|\lambda, \nu, \nu'\rangle$  for the representation  $\mathcal{SO}(2, 2)$  is characterized by

$$\mathcal{C}_{\mathfrak{so}(2,2)}^{(\mathcal{J}, \mathcal{L})} |\lambda, \nu, \nu'\rangle = \lambda(\lambda + 2) |\lambda, \nu, \nu'\rangle, \quad (8.a)$$

$$\mathcal{J}_0 |\lambda, \nu, \nu'\rangle = \nu |\lambda, \nu, \nu'\rangle, \quad (8.b)$$

$$\mathcal{L}_0 |\lambda, \nu, \nu'\rangle = \nu' |\lambda, \nu, \nu'\rangle, \quad (8.c)$$

and the basis  $|\lambda, \nu, \nu'\rangle$ , namely eigenfunctions, can be written explicitly as

$$\psi_{\nu, \nu'}(x) = \exp[i(\nu\phi + \nu'\chi)] \mathcal{R}(x), \quad (9)$$

It turns out from Eqs.(7-9) that the eigenvalues of the first Casimir operator are  $\lambda(\lambda + 2)$  where  $\lambda$  is connected with the eigenvalues  $j(j + 1)$  of the Casimir operator of  $\mathfrak{so}(2, 1)$  through the relation  $\lambda = 2j$ , while the second Casimir operator (7.b) has always zero eigenvalues [20]. Here  $\nu$  and  $\nu'$  are the eigenvalues of the compact generators  $\mathcal{J}_0$  and  $\mathcal{L}_0$ , respectively. The eigenfunction  $\mathcal{R}(x)$  is the physical wavefunction depending only in  $x$ , while  $\phi$  and  $\chi$  are auxiliary variables [19,20]. In terms of this realization, the Casimir operator (7.a) has the form

$$\begin{aligned} \mathcal{C}_{\mathfrak{so}(2,2)}^{(\mathcal{J}, \mathcal{L})} &= 2[-a\mathcal{J}_+\mathcal{J}_- + \mathcal{J}_0^2 - \mathcal{J}_0 - b\mathcal{L}_+\mathcal{L}_- + \mathcal{L}_0^2 - \mathcal{L}_0] \\ &= 4bh_2^2\partial_x^2 + 4bh_2(h_2' + 2g_2 - k_2)\partial_x + 4b(h_2g_2' + g_2^2 - k_2g_2) \\ &\quad + 2[1 - b(k_2^2 + f_2^2)](\mathcal{J}_0^2 + \mathcal{L}_0^2) - 8bf_2k_2\mathcal{J}_0\mathcal{L}_0. \end{aligned} \quad (10)$$

Now, inserting (9) into (10) taking into account (6), we get

$$\begin{aligned}
\mathcal{C}_{\text{so}(2,2)}^{(\mathcal{J},\mathcal{L})} = & 4 \frac{y^2}{y'^2} \partial_x^2 + 4 \frac{y}{y'} \left[ 2 \frac{g_2}{\sqrt{b}} - \frac{yy''}{y'^2} - \frac{2qy^2/\sqrt{b}}{1 - qy^2/\sqrt{b}} \right] \partial_x + 4 \left[ \frac{y}{y'} \frac{g_2'}{\sqrt{b}} \right. \\
& + \left( \frac{g_2}{\sqrt{b}} \right)^2 - \frac{1 + qy^2/\sqrt{b}}{1 - qy^2/\sqrt{b}} \frac{g_2}{\sqrt{b}} \left. \right] + 2 (\nu^2 + \nu'^2) \left[ 1 - \frac{b\delta^2 y}{(1 - qy^2/\sqrt{b})^2} \right. \\
& \left. - \left( \frac{1 + qy^2/\sqrt{b}}{1 - qy^2/\sqrt{b}} \right)^2 \right] - 8b\delta\sqrt{b}\nu\nu'y \frac{1 + qy^2/\sqrt{b}}{(1 - qy^2/\sqrt{b})^2}. \tag{11}
\end{aligned}$$

In the other hand, the general form of the Hamiltonians introduced by von Roos [2] for the spatially varying mass  $M(x) = m_0 m(x)$ , where  $m(x)$  is a dimensionless mass, read

$$\mathcal{H}_{VR} = \frac{1}{4} [m^\eta(x) p m^\epsilon(x) p m^\rho(x) + m^\rho(x) p m^\epsilon(x) p m^\eta(x)] + V(x), \tag{12}$$

where  $m_0 = 1$  and the restriction on the parameters  $\eta$ ,  $\epsilon$  and  $\rho$  checks the condition  $\eta + \epsilon + \rho = -1$ . Here  $p (\equiv -i\hbar\partial_x)$  is the momentum. In the natural units ( $\hbar = c = 1$ ), the Hamiltonian  $\mathcal{H}_{VR}$  becomes

$$\begin{aligned}
\mathcal{H}_{VR} = & -\frac{1}{4} \left[ \frac{2}{m} \partial_x^2 - 2 \frac{m'}{m} \partial_x - (1 + \epsilon) \frac{m''}{m^2} + 2 \{ \eta(\eta + \epsilon + 1) + \epsilon + 1 \} \frac{m'^2}{m^3} \right] \\
& + V(x). \tag{13}
\end{aligned}$$

By introducing the eigenfunctions [7]

$$\psi_\sigma(x) = 2\sigma m(x) \frac{y^2(x)}{y'^2(x)} \mathcal{R}(x), \tag{14}$$

where  $\sigma \in \mathbb{R}$ , letting now (13) acts on (14), and after some long and

straightforward algebra, we obtain

$$\begin{aligned}\mathcal{H}_{VR} = & -\sigma \frac{y^2}{y'^2} \partial_x^2 - \frac{\sigma y}{y'} \left[ 4 + \frac{m'y}{my'} - \frac{4yy''}{y'^2} \right] \partial_x + \frac{2\sigma y}{y'^2} \left[ 3y'' + \frac{yy'''}{y'} \right. \\ & \left. - \frac{3yy''^2}{y'^2} \right] + \frac{\sigma m'y^2}{my'^2} \left[ \frac{2\eta(yy'' - y'^2)}{yy'} - (1 + \eta)(\eta + \epsilon) \frac{m'}{m} \right] \\ & - 2\sigma + \frac{\sigma(\epsilon - 1)m''}{2m} \frac{y^2}{y'^2} + \frac{2\sigma my^2}{y'^2} V(x). \quad (15)\end{aligned}$$

$\mathcal{C}_{\mathfrak{so}(2,2)}^{(\mathcal{J},\mathcal{L})}$  and  $\mathcal{H}_{VR}$  are related to the undetermined function  $y(x)$  which results, as we will see in the next section, on a simple differential equation(s).

### 3 Spectrum-generating algebra of $\mathfrak{so}(2,2)$

The Schrödinger equations can be solved once equating them to the eigenvalues equation of the Casimir invariant operator of the  $\mathfrak{so}(2,2)$  algebra [26]

$$(\mathcal{H}_{VR} - E) \psi(x) = Z(x) \left( \mathcal{C}_{\mathfrak{so}(2,2)}^{(\mathcal{J},\mathcal{L})} - c \right) \psi(x) = 0, \quad (16)$$

where  $Z(x)$  is some function to be determined, and  $c$  is the eigenvalue of the Casimir operator. Now substituting (11) and (15) into (16) and comparing both sides we get

$$Z(x) = -\frac{\sigma}{4}, \quad (17.a)$$

$$\frac{g_2(x)}{\sqrt{b}} = \frac{2 - qy^2(x)/\sqrt{b}}{1 - qy^2(x)/\sqrt{b}} - \frac{3y(x)y''(x)}{2y'^2(x)} + \frac{m'(x)y(x)}{2m(x)y'(x)}. \quad (17.b)$$

Inserting  $g_2(x)$ ,  $g'_2(x)$  and  $g''_2(x)$  as defined in (17.b) into (16), taking into consideration (11) and (15), we end up with

$$V_{\text{eff}}^{(q,b)}(x) - E_{\nu,\nu',c} = \left[ \frac{\nu^2 + \nu'^2}{4} \frac{4q/\sqrt{b} + b\delta^2}{\left(1 - qy^2/\sqrt{b}\right)^2} + b\delta\sqrt{b}\nu\nu' \frac{1 + qy^2/\sqrt{b}}{y\left(1 - qy^2/\sqrt{b}\right)^2} + \frac{c}{8y^2} - \frac{q/\sqrt{b}}{2\left(1 - qy^2/\sqrt{b}\right)^2} \right] \frac{y'^2}{m} + \mathcal{V}_{\text{eff}}^{(\eta,\epsilon)}(x), \quad (18)$$

where by definition

$$\mathcal{V}_{\text{eff}}^{(\eta,\epsilon)}(x) = \frac{3}{8m} \frac{y''^2}{y'^2} - \frac{1}{4m} \frac{y'''}{y'} + \frac{m'^2}{8m^3} [(1 + 2\eta)^2 + 4\epsilon(1 + \eta)] - \frac{\epsilon m''}{4m^2}, \quad (19)$$

is called the effective potential related to the position-dependent mass and depends on the parameters  $\eta$  and  $\epsilon$ . In order to write suitably (18), a new change of parameters  $(\nu, \nu') \rightarrow (r, t)$  is introduced

$$\nu = \frac{1}{2} (\sqrt{r} + \sqrt{t}) \quad ; \quad \nu' = \frac{1}{2} (\sqrt{r} - \sqrt{t}), \quad (20)$$

which brings (18) to

$$V_{\text{eff}}^{(q,b)}(x) - E_{r,t,c} = \frac{r}{8} \left[ \frac{4q/\sqrt{b} + b\delta^2}{\left(1 - qy^2/\sqrt{b}\right)^2} + 2b\delta\sqrt{b} \frac{1 + qy^2/\sqrt{b}}{y\left(1 - qy^2/\sqrt{b}\right)^2} \right] \frac{y'^2}{m} + \frac{t}{8} \left[ \frac{4q/\sqrt{b} + b\delta^2}{\left(1 - qy^2/\sqrt{b}\right)^2} - 2b\delta\sqrt{b} \frac{1 + qy^2/\sqrt{b}}{y\left(1 - qy^2/\sqrt{b}\right)^2} \right] \frac{y'^2}{m} + \frac{c}{8y^2} \frac{y'^2}{m} - \frac{q/\sqrt{b}}{2\left(1 - qy^2/\sqrt{b}\right)^2} \frac{y'^2}{m} + \mathcal{V}_{\text{eff}}^{(\eta,\epsilon)}(x). \quad (21)$$



Without loss of generality, let us assume that the function  $y(x)$  is related to a certain *generating* function, namely,  $\mathfrak{S}(x)$  by

$$\frac{y'^2(x)}{m(x)} = \mathfrak{S}(x), \quad (22)$$

allowing us to rewrite (20) as

$$\begin{aligned} V_{\text{eff}}^{(q,b)}(x) - E_{r,t,c} &= \frac{r}{8}F(x)\mathfrak{S}(x) + \frac{t}{8}G(x)\mathfrak{S}(x) + \frac{c}{8y^2(x)}\mathfrak{S}(x) \\ &\quad - \frac{q/\sqrt{b}}{2\left(1 - qy^2(x)/\sqrt{b}\right)^2}\mathfrak{S}(x) + \mathcal{V}_{\text{eff}}^{(\eta,\epsilon)}(x), \end{aligned} \quad (23)$$

where  $F(x)$  and  $G(x)$  are defined by

$$F(x) = \frac{4q/\sqrt{b} + b\delta^2}{\left(1 - qy^2(x)/\sqrt{b}\right)^2} + 2b\delta\sqrt{b}\frac{1 + qy^2(x)/\sqrt{b}}{y(x)\left(1 - qy^2(x)/\sqrt{b}\right)^2} \quad (24.a)$$

$$G(x) = \frac{4q/\sqrt{b} + b\delta^2}{\left(1 - qy^2(x)/\sqrt{b}\right)^2} - 2b\delta\sqrt{b}\frac{1 + qy^2(x)/\sqrt{b}}{y(x)\left(1 - qy^2(x)/\sqrt{b}\right)^2} \quad (24.b)$$

Equation (23) is our main result, it is the key formula for generalized potentials. Indeed it opens up ways to recover a wide kind of exactly solvable potentials defined by Natanzon [27]. Observing that the energy  $E_{r,t,c}$  on the left-hand side of (23) represents a constant term and, therefore, must be equated to a certain constant term in the right-hand side which has led to a simple differential equation for  $y(x)$ . On the other hand, we can assume that the energy term  $E_{r,t,c}$  can be expressed in terms of the coefficients  $r$ ,  $t$  and  $c$  which required that both functions  $y(x)$  and  $\mathfrak{S}(x)$  be independent of  $E_{r,t,c}$ .

On these settings, let us perform a formal derivative of (23) with respect of  $E_{r,t,c}$  and which henceforth will be quoted  $E$ , which leads to the expression of the generating function  $\mathfrak{S}(x)$  given by

$$\mathfrak{S}(x) = \frac{-1}{\frac{F(x)}{8}\partial_E r + \frac{G(x)}{8}\partial_E t + \frac{1}{8y^2(x)}\partial_E c}. \quad (25)$$

Let the generating function  $\mathfrak{S}(x)$  be positive, we assume that the derivatives of the coefficients  $r$ ,  $t$  and  $c$  with respect of  $E$  in (25) are constant, which requires that the coefficients are linear with respect to  $E$  [26]. In terms of these settings, the coefficients become

$$\partial_E r = -r_0 \implies r(E) = -r_0 E + a_r, \quad (26.a)$$

$$\partial_E t = -t_0 \implies t(E) = -t_0 E + a_t, \quad (26.b)$$

$$\partial_E c = -c_0 \implies c(E) = -c_0 E + a_c, \quad (26.c)$$

where  $r_0$ ,  $t_0$ ,  $c_0$ ,  $a_r$ ,  $a_t$  and  $a_c$  are six real parameters. A straightforward algebraic manipulation permits to recast the generating function  $\mathfrak{S}(x)$  through

$$\begin{aligned} \mathfrak{S}(x) &\equiv \frac{y^2(x)}{m(x)} \\ &= \frac{8y^2(x)}{r_0 F(x) y^2(x) + t_0 G(x) y^2(x) + c_0}. \end{aligned} \quad (27)$$

One can see that the first three parameters  $(r_0, t_0, c_0)$  govern completely the behavior of the function  $y(x)$ , while, as we will see later, the last three parameters  $(a_r, a_t, a_c)$  determine the shape of the energy eigenvalues.

## 4 Construction of exactly solvable potentials via SGA

Here we illustrate the procedure by which a wide kind of exactly solvable potentials belonging to the Natanzon and Natanzon confluent potentials can be recovered using the SGA scheme. All these potentials fall into two cases (classes) and are referred to (i) the appropriate-parameter choices of  $(q, b)$  and  $(r_0, t_0, c_0)$ , and (ii) the function  $y(x)$ . These parameter sets can be determined arbitrary, while  $y(x)$  can be obtained, after integration, from the restriction (27). Among those choices, it is found that the Natanzon confluent potentials [27] are generated when the constraints  $q = 0$  and  $b = 1$  are fulfilled, while the Natanzon potentials can be constructed from  $q = 1$  and  $b = 1$ .

## 4.1 Case A : $q = 0$ , $b = 1$

In this subsection, the spectrum-generating algebra are explained in full details. The functions  $F(x)$  and  $G(x)$  are given by

$$F(x) = \frac{\delta^2 y(x) + 2\delta}{y(x)} \quad ; \quad G(x) = \frac{\delta^2 y(x) - 2\delta}{y(x)}, \quad (28)$$

and thus, from (27), the generating function  $\mathfrak{S}(x)$  becomes

$$\begin{aligned} \mathfrak{S}(x) &= \frac{y'^2(x)}{m(x)} \\ &= \frac{8y^2(x)}{c_0 + 2\delta(r_0 - t_0)y(x) + \delta^2(r_0 + t_0)y(x)}. \end{aligned} \quad (29)$$

The Morse, Harmonic oscillator and Coulomb-like potentials can be deduced from the generating function (29) and we will name them : exponential, quadratic and linear solutions, respectively [7,21,27].

### 4.1.1 Exponential solution : Morse-like potential

This solution corresponds to the appropriate-parameter choices  $c_0 \neq 0$  and  $r_0 - t_0 = r_0 + t_0 = 0$ , implying that  $r_0 = t_0 = 0$ . Consequently, the generating function  $\mathfrak{S}(x)$  is then determined from the differential equation  $c_0 y'^2(x) - 8m(x)y^2(x) = 0$ , where their solutions take the form

$$y(x) = \exp[-\alpha\mu(x)], \quad (30)$$

where  $\alpha = 2\sqrt{\frac{2}{c_0}}$  and the auxiliary function  $\mu(x) = \int^x du \sqrt{m(u)}$  is defined, due to  $m(x)$ , as dimensionless mass integral. Substituting (26) and (30) into (23), taking into consideration the restrictions concerning  $r_0$ ,  $t_0$  and  $c_0$  referred hereabove, we get

$$\begin{aligned} V_{\text{eff}}^{(0,1)}(x) - E &= \frac{\delta^2 \alpha^2}{8} (a_r + a_t) e^{-2\alpha\mu(x)} + \frac{\delta \alpha^2}{4} (a_r - a_t) e^{-\alpha\mu(x)} \\ &\quad + \frac{\alpha^2}{8} (a_c - c_0 E + 1) + \mathcal{U}_{\text{eff}}^{(\eta, \epsilon)}(x), \end{aligned} \quad (31)$$

where the effective potential  $\mathcal{U}_{\text{eff}}^{(\eta,\epsilon)}(x) \equiv \mathcal{V}_{\text{eff}}^{(\eta,\epsilon)}(x) - \frac{\alpha^2}{8}$  reads as

$$\mathcal{U}_{\text{eff}}^{(\eta,\epsilon)}(x) = \left[ 2\eta^2 + 2\eta(1 + \epsilon) + \frac{3\epsilon}{2} + \frac{7}{8} \right] \frac{\mu''^2(x)}{\mu'^4(x)} - \frac{1 + 2\epsilon}{4} \frac{\mu'''(x)}{\mu'^3(x)}. \quad (32)$$

It is obvious that we recognize in (31) the Morse effective-like potential solely if the  $a_r \neq \pm a_t$  constraint holds.

The energy eigenvalues can be deduced from (31) by equating the energy term in the left-hand side with the constant term in the right-hand side. Indeed, taking into account  $\alpha = 2\sqrt{\frac{2}{c_0}}$  and comparing both sides, we obtain

$$-E = \frac{\alpha^2}{8} (a_c - c_0 E + 1), \quad (33)$$

leading to the identification  $a_c = -1$ . In the other hand, it is well-known that the eigenvalues of the Casimir operator associated to  $\mathfrak{so}(2, 2)$  algebra are  $\lambda(\lambda + 2)$ , thus the energy eigenvalues associated with the Morse potential can be deduced using (26.c), i.e.

$$\begin{aligned} c(E) &= -c_0 E + a_c \\ &= -c_0 E - 1 \\ &\equiv \lambda(\lambda + 2), \end{aligned} \quad (34)$$

where finally the energy can be expressed in terms of  $\alpha$  and  $\lambda$  as

$$E_\lambda = -\frac{\alpha^2}{8} (1 + \lambda)^2. \quad (35)$$

Since we are dealing with the Morse-type potential, let us choose a general parameters  $A$  and  $B$  such that  $\begin{bmatrix} a_r \\ a_t \end{bmatrix} = 2B (\mp 2A \mp \alpha + \frac{2B}{\alpha^2 \delta^2})$  where  $a_r(a_t)$  agrees with upper (lower) sign, respectively. Therefore the Morse potential as defined in (31) can be written as

$$V_{\text{M}}^{(0,1)}(x) = B^2 \exp[-2\alpha\mu(x)] - B(2A + \alpha) \exp[-\alpha\mu(x)]. \quad (36)$$

#### 4.1.2 Quadratic solution : $3\mathcal{D}$ –Harmonic oscillator-like potential

The quadratic solution depends on the following appropriate-parameter choices  $c_0 = 0$ ,  $r_0 = -t_0$  and  $r_0 \neq t_0$ . The generating function is reduced to the simple differential equation  $\delta r_0 y'^2(x) - 2m(x)y(x) = 0$  yielding the solution

$$y(x) = \frac{\alpha}{4} \mu^2(x), \quad (37)$$

with  $\alpha = \frac{2}{\delta r_0}$ . Therefore, Eq.(26) becomes

$$\begin{aligned} V_{\text{eff}}^{(0,1)}(x) - E &= \frac{\delta^2 \alpha^2}{32} (a_r + a_t) \mu^2(x) + \frac{a_c + \frac{3}{4}}{2\mu^2(x)} - \frac{\delta \alpha r_0 E}{2} \\ &+ (a_r - a_t) \frac{\delta \alpha}{4} + \mathcal{V}_{\text{eff}}^{(\eta, \epsilon)}(x), \end{aligned} \quad (38)$$

where we recognize the  $3\mathcal{D}$ –harmonic oscillator-like potential.

The procedure to generate the energy eigenvalues corresponding to the  $3\mathcal{D}$ –Harmonic oscillator is exactly as before. Equating, in (38), the energy term in the left-hand side with the constant term in the right-hand side, taking into account the restrictions  $\alpha = \frac{2}{\delta r_0}$ ,  $c_0 = 0$  and  $r_0 = -t_0$ , we obtain

$$-\left(1 - \frac{r_0 \delta \alpha}{2}\right) E = (a_r - a_t) \frac{\delta \alpha}{4} = 0, \quad (39)$$

which results to identify that  $a_r = a_t$ . Dealing with the  $3\mathcal{D}$ –Harmonic oscillator potential, let us assume  $a_r = a_t = 1$  and  $\delta \alpha = 2\omega$  implying that  $r_0 = -t_0 = \omega^{-1}$ . By combining (26.a) to (26.b), we can deduce the analytic expression of the energy

$$E_{\nu, \nu'} = \delta \alpha \nu \nu' \implies E_n = 2\omega n. \quad (40)$$

where  $n = \nu \nu'$  is the *reduced* quantum number for the  $3\mathcal{D}$ –Harmonic oscillator under the  $\mathfrak{so}(2, 2)$  algebra. Knowing from (26.c) that  $c(E) \equiv a_c = \lambda(\lambda + 2)$ , the potential in (38) can be reduced to

$$V_{\text{H.O}}^{(0,1)}(x) = \frac{1}{4} \omega^2 \mu^2(x) + \frac{3 + 4\lambda(\lambda + 2)}{8\mu^2(x)}. \quad (41)$$

### 4.1.3 Linear solution : $3\mathcal{D}$ –Coulomb-like potential

The linear solution corresponds to the specific-parameter choices  $c_0 = 0$ ,  $r_0 = t_0$  and  $r_0 \neq -t_0$ . The generating function is reduced to the following differential equation  $\delta^2 r_0 y'^2(x) - 4m(x) = 0$ , from where the solution is given by  $y(x) = \alpha\mu(x)$  with  $\alpha = \frac{2}{\delta\sqrt{r_0}}$ . The effective potential reads as

$$V_{\text{eff}}^{(0,1)}(x) - E = \frac{\delta\alpha(a_r - a_t)}{2\mu(x)} + \frac{a_c}{8\mu^2(x)} - \frac{r_0\delta^2\alpha^2 E}{4} + \frac{\delta^2\alpha^2}{8}(a_r + a_t) + \mathcal{V}_{\text{eff}}^{(\eta,\epsilon)}(x), \quad (42)$$

where we recognize here the three-dimensional Coulomb-like potential.

The energy eigenvalues accompanying the potential (42) can be deduced once equating the constant terms involving in (42), and after some algebraic manipulations we obtain

$$a_r = -a_t, \quad (43.a)$$

$$c(E) \equiv a_c = \lambda(\lambda + 2), \quad (43.b)$$

and combining, once again, (26.a) to (26.b) we deduce the analytic expression of the energy related to the  $3\mathcal{D}$ –Coulomb potential

$$E_{\nu,\nu'} = -\frac{\nu^2 + \nu'^2}{r_0}, \quad (44)$$

where  $r_0 = \frac{4}{\delta^2\alpha^2}$ . Since we are dealing with three-dimensional Coulomb potential, we will assume that  $Ze^2 = \delta\alpha\nu\nu'$ , then Eq.(44) becomes

$$E_{\nu,\nu'} = -\frac{Z^2e^4}{4} \left[ \frac{1}{\nu^2} + \frac{1}{\nu'^2} \right] \implies E_{\mathcal{N}} = -\left[ \frac{Ze^2}{2\mathcal{N}} \right]^2, \quad (45)$$

with  $\mathcal{N} = \frac{\nu\nu'}{\sqrt{\nu^2 + \nu'^2}}$  is the reduced quantum number for the  $3\mathcal{D}$ –Coulomb potential under  $\mathfrak{so}(2, 2)$  algebra. Knowing from (26.c) that  $c(E) \equiv a_c = \lambda(\lambda + 2)$  and  $a_r = -a_t = -Ze^2$  the potential in (42) can be written as

$$V_{\text{Cb}}^{(0,1)}(x) = -\frac{Ze^2}{\mu(x)} + \frac{\lambda(\lambda + 2)}{8\mu^2(x)}. \quad (46)$$

## 4.2 Case B : $q = 1, b = 1$

In this case, the functions  $F(x)$  and  $G(x)$  become, from Eqs.(24)

$$F(x) = \frac{(4 + \delta^2) y(x) + 2\delta (1 + y^2(x))}{y(x) (1 - y^2(x))^2}, \quad (47.a)$$

$$G(x) = \frac{(4 + \delta^2) y(x) - 2\delta (1 + y^2(x))}{y(x) (1 - y^2(x))^2}, \quad (47.b)$$

and the generating function  $\mathfrak{S}(x)$  is given by

$$\begin{aligned} \mathfrak{S}(x) &\equiv \frac{y'^2(x)}{m(x)} \\ &= \frac{8y^2(x) (1 - y^2(x))^2}{c_0 (1 - y^2(x))^4 + (r_0 + t_0) (4 + \delta^2) y^2(x) + 2\delta (r_0 - t_0) y(x) (1 - y^2(x))} \end{aligned} \quad (48)$$

Two particular cases arise and are associated to the appropriate-parameters choices of  $r_0$ ,  $t_0$ , and  $c_0$ . Thus a wide kind of exactly solvable potentials, belonging to the Natanzon potentials, can be deduced such as the generalized Pöschl-Teller, Pöschl-Teller, Scarf II, Eckart, Hulthén and Rosen-Morse-like potentials as well as their trigonometric versions [7,21,27]. Here we present our results without giving the details of our calculations which are straightforward and are exactly as before.

### 4.2.1 Case B-I : $r_0 = t_0 = 0, c_0 \neq 0$

**The generalized Pöschl-Teller potential and its eigenvalues.** Here the generating function (48) has as solution

$$y(x) = \exp[-\alpha\mu(x)], \quad (49)$$

with  $\alpha = 2\sqrt{\frac{2}{c_0}}$ . Consequently, the deduced potential and its corresponding energy eigenvalues are given, respectively, by

$$\begin{aligned} V_{\text{GPT}}^{(1,1)}(x) &= \frac{\alpha^2}{32} [(4 + \delta^2) (a_r + a_t) - 4] \operatorname{cosech}^2 \alpha\mu(x) \\ &\quad + \frac{\delta\alpha^2 (a_r - a_t)}{8} \operatorname{cosech} \alpha\mu(x) \coth \alpha\mu(x), \end{aligned} \quad (50)$$

$$E_\lambda = -\frac{\alpha^2}{8} (1 + \lambda)^2. \quad (51)$$

If the restriction  $a_r \neq \pm a_t$  holds, then we recognize here the generalized Pöschl-Teller-like potential. The extension towards its trigonometric version is possible once the substitution  $\alpha \rightarrow i\alpha$  is made leading to

$$\begin{aligned} V_{\text{GPT}}^{(1,1)}(x) = & \frac{\alpha^2}{32} [(4 + \delta^2)(a_r + a_t) - 4] \csc^2 \alpha\mu(x) \\ & + \frac{\delta\alpha^2(a_r - a_t)}{8} \csc \alpha\mu(x) \cot \alpha\mu(x), \end{aligned} \quad (52)$$

$$E_\lambda = \frac{\alpha^2}{8} (1 + \lambda)^2. \quad (53)$$

**The Pöschl-Teller potential and its eigenvalues.** In order to deduce the Pöschl-Teller potential, let us multiply the exponent of (49) by 2, i.e.  $y(x) = \exp[-2\alpha\mu(x)]$ , that implies to redefine a new parameter  $\alpha$  as  $\alpha \rightarrow \alpha = \sqrt{\frac{2}{c_0}}$ . The potential becomes

$$\begin{aligned} V_{\text{PT}}^{(1,1)}(x) = & -\frac{\alpha^2}{32} [(\delta + 2)^2 a_t + (\delta - 2)^2 a_r - 4] \operatorname{sech}^2 \alpha\mu(x) \\ & + \frac{\alpha^2}{32} [(\delta + 2)^2 a_r + (\delta - 2)^2 a_t - 4] \operatorname{cosech}^2 \alpha\mu(x), \end{aligned} \quad (54)$$

and its accompanying energy eigenvalues are given by

$$E_\lambda = -\frac{\alpha^2}{2} (1 + \lambda)^2. \quad (55)$$

We have generated here the hyperbolic Pöschl-Teller-like potential. Its trigonometric version is obtained once the substitution  $\alpha \rightarrow i\alpha$  is carried out, given by

$$\begin{aligned} V_{\text{PT}}^{(1,1)}(x) = & \frac{\alpha^2}{32} [(\delta + 2)^2 a_t + (\delta - 2)^2 a_r - 4] \sec^2 \alpha\mu(x) \\ & + \frac{\alpha^2}{32} [(\delta + 2)^2 a_r + (\delta - 2)^2 a_t - 4] \csc^2 \alpha\mu(x), \end{aligned} \quad (56)$$



$$E_\lambda = \frac{\alpha^2}{2} (1 + \lambda)^2. \quad (57)$$

**The Scarf II potential and its eigenvalues.** The last potential obtainable in this category relates to the choice  $y(x) = \exp[-\alpha\mu(x) + \frac{i\pi}{2}]$ . Due to the shape of  $y(x)$ , this potential is called *isospectral* to the generalized Pöschl-Teller potential because they share the same energy eigenvalues. The potential and its corresponding energy eigenvalues read, respectively

$$\begin{aligned} V_{\text{SC}}^{(1,1)}(x) = & -\frac{\alpha^2}{32} [(4 + \delta^2)(a_r + a_t) - 4] \operatorname{sech}^2 \alpha\mu(x) \\ & + i \frac{\delta\alpha^2(a_r - a_t)}{8} \operatorname{sech} \alpha\mu(x) \tanh \alpha\mu(x), \end{aligned} \quad (58)$$

$$E_\lambda = -\frac{\alpha^2}{8} (1 + \lambda)^2. \quad (59)$$

We generate here the hyperbolic  $\mathcal{PT}$ -symmetric Scarf-like potential. One can see that the energy eigenvalues (51) and (59) coincide, thus it is then obvious that the generalized Pöschl-Teller and the Scarf II potentials are isospectral. The non- $\mathcal{PT}$ -symmetric potential, corresponding to the trigonometric case, is obtained once the substitution  $\alpha \rightarrow i\alpha$  is made, given

$$\begin{aligned} V_{\text{SC}}^{(1,1)}(x) = & \frac{\alpha^2}{32} [(4 + \delta^2)(a_r + a_t) - 4] \sec^2 \alpha\mu(x) \\ & + i \frac{\delta\alpha^2(a_r - a_t)}{8} \sec \alpha\mu(x) \tan \alpha\mu(x), \end{aligned} \quad (60)$$

$$E_\lambda = \frac{\alpha^2}{8} (1 + \lambda)^2. \quad (61)$$

#### 4.2.2 Case B-II : $r_0 = t_0 \neq 0$ , $c_0 = 0$

Here, the generating function (48) is reduced to

$$\mathfrak{S}(x) \equiv \frac{y'^2(x)}{m(x)} = \frac{4p(1 - y^2(x))^2}{r_0(4 + \delta^2)}, \quad (62)$$

where the class of solutions for the differential equation (62) dependent on the arbitrary parameter  $p = 1, 2, \dots$  are given through

$$y_p(x) = \tanh \frac{\alpha \mu(x)}{2p}, \quad (63)$$

with  $\alpha = \frac{4p}{\sqrt{r_0(4+\delta^2)}}$ .

**The Eckart potential and its eigenvalues.** For  $p = 1$  and taking into consideration the hyperbolic identity

$$\frac{1 + \tanh^2 a}{\tanh \frac{a}{2}} = 2 \coth a, \quad (64)$$

one deduces the potential and its accompanying energy eigenvalues, respectively

$$V_{\text{Eck}}^{(1,1,1)}(x) = -\frac{\delta \alpha^2 a_t}{4} \coth \alpha \mu(x) + \frac{\alpha^2 \lambda (\lambda + 2)}{8} \text{cosech}^2 \alpha \mu(x), \quad (65)$$

$$E_{\nu, \nu'} = -\frac{\alpha^2}{16} (4 + \delta^2) (\nu^2 + \nu'^2). \quad (66)$$

We recognize here the Eckart-like potential. In the other hand its trigonometric version is obtained once both substitutions  $\alpha \rightarrow i\alpha$  and  $\delta \rightarrow -i\delta$  are made, given

$$V_{\text{Eck}}^{(1,1,1)}(x) = -\frac{\delta \alpha^2 a_t}{4} \cot \alpha \mu(x) + \frac{\alpha^2 \lambda (\lambda + 2)}{8} \csc^2 \alpha \mu(x), \quad (67)$$

$$E_{\nu, \nu'} = \frac{\alpha^2}{16} (4 - \delta^2) (\nu^2 + \nu'^2). \quad (68)$$

**The Hulthén potential and its eigenvalues.** For  $p = 2$ , we generate the following potential

$$V_{\text{Hult}}^{(1,1,2)}(x) = \frac{\alpha^2}{4} \left[ \lambda(\lambda + 2) + \frac{\delta\nu\nu'}{2} \right] \frac{e^{-\alpha\mu(x)}}{2(1 - e^{-\alpha\mu(x)})} + \frac{\alpha^2}{4} \lambda(\lambda + 2) \frac{e^{-2\alpha\mu(x)}}{2(1 - e^{-\alpha\mu(x)})^2}, \quad (69)$$

and its associated energy eigenvalues are given by

$$E_{\nu,\nu'} = -\frac{\alpha^2}{16} \left[ (4 + \delta^2)(\nu^2 + \nu'^2) + 4\delta\nu\nu' \right]. \quad (70)$$

The expressions (69) and (70) are known in the literature as being the Hulthén potential and energy eigenvalues, respectively.

**The Rosen-Morse potential and its eigenvalues.** Now we are interested in the isospectral potential which is associated to the Eckart potential with the coefficient  $\frac{i\pi}{4}$ . Consequently, with  $p = 1$ , the generating function becomes  $y(x) = \tanh \left[ \frac{\alpha\mu(x)}{2} + \frac{i\pi}{4} \right]$ . In this respect, the use of the hyperbolic identity

$$\tanh(a + ib) = \frac{\tanh a + i \tanh b}{1 + i \tanh a \tanh b}, \quad (71)$$

led, after a long calculation, to deduce the following potential

$$V_{\text{RM}}^{(1,1,1)}(x) = \frac{\delta\alpha^2}{8} (a_r - a_t) \tanh \alpha\mu(x) - \frac{\alpha^2}{8} \lambda(\lambda + 2) \text{sech}^2 \alpha\mu(x), \quad (72)$$

where their accompanying energy eigenvalues are given

$$E_{\nu,\nu'} = -\frac{\alpha^2}{16} (4 + \delta^2)(\nu^2 + \nu'^2). \quad (73)$$

The potential (72) and the energy eigenvalues (73) are those of the Rosen-Morse type. Indeed one can see that the Eckart and the Rosen-Morse potentials are isospectral, i.e. sharing the same energy eigenvalues.

## 5 Conclusion

In this article, we have discussed the general differential realization of the potential group  $\mathcal{SO}(2, 2)$  and we have solved the Schrödinger equation by identifying it to the eigenvalues equation of the Casimir invariant operator(s) of this algebra. We have analyzed the role of the two  $\mathfrak{so}(2, 1)$  subalgebras and point out their importance by means of spectrum-generating algebra techniques for a wide kind of potentials, for which an analytic solution to the bound-state problem have been deduced; i.e. all potentials defined by Natanzon [27]. We have constructed the case A corresponding to  $q = 0$  and  $b = 1$  in full detail thus generating the Morse, Harmonic oscillator and Coulomb-like potentials, while the case B with  $q = 1$  and  $b = 1$  generates the generalized Pöschl-Teller, Pöschl-Teller, Scarf II, Eckart, Hulthén and Rosen-Morse-like potentials as well as their trigonometric versions. In terms of these settings, it becomes clear that the former (case A) corresponds to the Natanzon confluent potentials including the confluent hypergeometric functions as wavefunctions, while the latter (case B) coincides with the Natanzon potentials and their wavefunctions include hypergeometric functions.

In the light of all these, it has been shown that the exact solution of the position-dependent effective mass Schrödinger equation leads to a general solution of the Natanzon potentials, which are independent of the choice of the parameters  $\eta$ ,  $\epsilon$  and  $\rho$ .

## References

- [1] D. J. BenDaniel, C. B. Duke, Phys. Rev. **152** (1966) 683.
- [2] O. von Roos, Phys. Rev. B **27** (1983) 7547.
- [3] L. Dekar, I. Chetouani, F. T. Hammann, J. Phys. A : Math. Gen. **39** (1998) 2551;  
L. Dekar, I. Chetouani, F. T. Hammann, Phys. Rev. A **59** (1999) 107.

- [4] B. Gönül, O. Özer, B. Gönül, F. Üzgün, Mod. Phys. Lett. A **17** (2002) 2453.
- [5] A. D. Alhaidari, Phys. Rev. A **65** (2002) 042109;  
A. D. Alhaidari, Phys. Rev. A **66** (2002) 189901;  
A. D. Alhaidari, Int. J. Theo. Phys. **42** (2003) 2999.
- [6] B. Roy, P. Roy, J. Phy. A : Math. Gen. **35** (2002) 3961;  
B. Roy, P. Roy, Phys. Lett. A **340** (2005) 70.
- [7] R. Koç, M. Koca, J. Phys. A : Math. Gen. **36** (2003) 8105.
- [8] B. Bagchi, A. Banerjee, C. Quesne, V. M. Tkachuk, J. Phys. A : Math. Gen. **38** (2005) 2929.
- [9] J. Yu, S. H. Dong, G. H. Sun, Phys. Lett. A **322** (2004) 290.
- [10] C. Y. Cai, Z. Z. Ren, G. X. Ju, Commn. Theor. Phys. **43** (2005) 1019.
- [11] G. Harrison, "Quantum Wells, Wires and Dots", Academic Press, New york, 1993.
- [12] M. Barranco, M. Pi, S. M. Gatica, E. S. Hernandez, J. Navarro, Phys. Rev. B **56** (1997) 8997.
- [13] F. Arias de Saavedra, J. Boronat, A. Polls, A. Fabroccini, Phys. Rev. B **50** (1997) 4248.
- [14] C. Weisbach, B. Vinter, "Quantum Semiconductor Heterostructures", Academic Press, New york, 1993.
- [15] G. Bastard, "Wave Mechanics Applied to Heterostructures", les Ulis, les éditions de Physique, 1989.
- [16] F. Cooper, A. Khare, U. Sukhatme, Phys. Rep. **251** (1995) 267.
- [17] Y. Alhassid, F. Gürsey, F. Iachello, Phys. Rev. Lett. **50** (1983) 873;  
Y. Alhassid, F. Gürsey, F. Iachello, Ann. Phys. **148** (1983) 346.

- [18] Y. Alhassid, F. Iachello, J. Wu, Phys. Rev. Lett. **56** (1986) 271.
- [19] J. Wu, Y. Alhassid, J. Math. Phys. **31** (1990) 557.  
J. Wu, Y. Alhassid, F. Gürsey, Ann. Phys. **196** (1989) 163.
- [20] G. Lévai, F. Cannata, A. Ventura, J. Phys. A : Math. Gen. **35** (2002) 5041.
- [21] G. Lévai, J. Phys. A : Math. Gen. **27** (1994) 3809.
- [22] A. O. Barut, R. Raczka, "Theory of Group Representation and Applications", PWN, Warsaw, 1977.
- [23] P. Cordero, G. C. Ghirardi, Nuovo Cimento A **2** (1971) 817;  
P. Cordero, G. C. Ghirardi, Fortsch. Phys. **20** (1972) 105.
- [24] P. Cordero, S. Hojman, P. Furlan, G. C. Ghirardi, Nuovo Cimento A **3** (1971) 807.
- [25] P. Cordero, S. Salomó, J. Phys. A : Math. Gen. **24** (1991) 5299.
- [26] P. Cordero, S. Salomó, Found. Phys. **23** (1993) 675.
- [27] G. A. Natanzon, Teor. Mat. Fiz. **38** (1979) 146.
- [28] S.-A. Yahiaoui, S. Hattou, M. Bentaiba, Ann. Phys. **322** (2007) 2433.